Ronald L. Pryor, Ph.D.

Sets: Characteristically Speaking

A constant challenge in Systems Science, as with any of the sciences, is to discover more efficient and effective ways to model the real world. After spending many years teaching discrete mathematics to computer science students and trying to fit the topic of elementary set theory to the "discrete" nature of the subject, I found that using the characteristic function to define sets provided that proper fit and afforded a more robust set of modeling tools to my students.

Short Biography:

Dr. Pryor has been an educator for almost fifty years teaching a wide range of students from middle school and high school to graduate school. He received his BA and MS in Mathematics from Wilkes College and a Ph.D. in Systems Science from SUNY at Binghamton. While at Binghamton he contributed to Some Non-Biological Applications of L-Systems, by Goel and Rozehnal, International Journal of General Systems, Vol. 18, pp. 321-405 and to the text Uncertainty and Information: Foundations of Generalized Information Theory, John Wiley, Hoboken, NJ by George Klir.

Dr. Pryor is currently a member of the Department of Mathematics and Computer Science at Wilkes University and teaches courses in both Mathematics and Computer Science. His current areas of interest include Cybernetics and Systems Science, Generalized Information Theory, Generalized Measure Theory, and Uncertainty-Based Information, Fuzzy Sets and Fuzzy Systems, Soft Computing, Computational Intelligence and Agent-Based Modeling.

Rita, feel free to edit if needed.

DeMorgan (1)

\[ \overline{A \cup B} = \overline{A} \cap \overline{B} \]
\[ \overline{A \cup B} = \chi_A(x) + \chi_B(x) - \chi_A(x) \cdot \chi_B(x) \]
\[ = \chi_A(x) + \chi_B(x) - \left(1 - \chi_A(x)\right) \cdot \left(1 - \chi_B(x)\right) \]
\[ = \chi_A(x) + \chi_B(x) - \left(1 - \chi_A(x) - \chi_B(x) + \chi_A(x) \cdot \chi_B(x)\right) \]
\[ = 1 - \chi_A(x) + 1 - \chi_B(x) - 1 + \chi_A(x) + \chi_B(x) - \chi_A(x) \cdot \chi_B(x) \]
\[ = 1 - \chi_A(x) \cdot \chi_B(x) \]
\[ = \chi_{\overline{A} \cap \overline{B}}(x) \]
\[ = \overline{A \cap B} \]
<p>| | | | | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( A )</td>
<td>( B )</td>
<td>( \overline{A} )</td>
<td>( \overline{B} )</td>
<td>( \overline{A} \cup \overline{B} )</td>
<td>( A \cap B )</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1 + 1 - 1 \times 1 = 1</td>
<td>0 \times 0 = 0</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>1 + 0 - 1 \times 0 = 1</td>
<td>0 \times 1 = 0</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0 + 1 - 0 \times 1 = 1</td>
<td>1 \times 0 = 0</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0 + 0 - 0 \times 0 = 0</td>
<td>1 \times 1 = 1</td>
</tr>
</tbody>
</table>

De Morgan(2)

\[
\overline{A \cup B} = \overline{A} \cap \overline{B} \\
\overline{A \cap B} = 1 - \chi_{A \cup B}(x) \\
= 1 - \left( \chi_A(x) + \chi_B(x) - \chi_A(x) \cdot \chi_B(x) \right) \\
= 1 - \chi_A(x) - \chi_B(x) + \chi_A(x) \cdot \chi_B(x) \\
= \left(1 - \chi_A(x)\right) \left(1 - \chi_B(x)\right) \\
= \chi_{\overline{A} \cap \overline{B}}(x) \\
= \overline{A \cap B}
\]

Inductive Case for De Morgan's Law (Crisp Sets)

\[
\overline{A_1 \cap A_2 \cap \ldots \cap A_n} = \overline{A_1} \cup \overline{A_2} \cup \ldots \cup \overline{A_n}
\]

Assume \( \overline{A_1 \cap A_2 \cap \ldots \cap A_k} = \overline{A_1} \cup \overline{A_2} \cup \ldots \cup \overline{A_k} \)

show \( \overline{A_1 \cap A_2 \cap \ldots \cap A_{k+1}} = \overline{A_1} \cup \overline{A_2} \cup \ldots \cup \overline{A_k} \)

Define \( B = A_1 \cap A_2 \cap \ldots A_k \)

\[
\overline{A_1 \cap A_2 \cap \ldots A_{k+1}} = \overline{B \cap A_{k+1}} \\
= \overline{B} \cup \overline{A_{k+1}} \\
= \overline{A_1 \cap A_2 \cap \ldots \overline{A_k} \cup \overline{A_{k+1}}} \\
= \overline{A_1} \cup \overline{A_2} \cup \ldots \overline{A_k} \cup \overline{A_{k+1}}
\]
**Law of Excluded Middle (Crisp)**

\[ A \cup \overline{A} = X \]
\[ A \cup \overline{A} = \mathcal{X}_A(x) + \mathcal{X}_{\overline{X}}(x) - \mathcal{X}_A(x) \cdot \mathcal{X}_{\overline{X}}(x) \]
\[ = \mathcal{X}_A(x) + (1 - \mathcal{X}_A(x)) \cdot (1 - \mathcal{X}_A(x)) \]
\[ = \mathcal{X}_A(x) + 1 - \mathcal{X}_A(x) - \mathcal{X}_A(x) + \mathcal{X}_A(x) \cdot \mathcal{X}_A(x) \]
\[ = \mathcal{X}_A(x) + 1 - \mathcal{X}_A(x) - \mathcal{X}_A(x) + \mathcal{X}_A(x) \]
\[ = 1 \]
\[ = X \]

**Law of Contradiction (Crisp)**

\[ A \cap \overline{A} = \emptyset \]
\[ A \cap \overline{A} = \mathcal{X}_A(x) \cdot \mathcal{X}_{\overline{X}}(x) \]
\[ = \mathcal{X}_A(x) \cdot (1 - \mathcal{X}_A(x)) \]
\[ = \mathcal{X}_A(x) - \mathcal{X}_A(x) \cdot \mathcal{X}_A(x) \]
\[ = \mathcal{X}_A(x) - \mathcal{X}_A(x) \]
\[ = 0 \]
\[ = \emptyset \]

**Law of Excluded Middle (Fuzzy)**

\[ A \cup \overline{A} \neq X \]

Let \( \mu_A(x) = 0.4 \), \( \therefore \mu_{\overline{X}}(x) = 0.6 \)

so \( \mu_{A \cup B}(x) = \max[\mu_A(x), \mu_{\overline{X}}(x)] \)
\[ = \max[0.4, 0.6] \]
\[ = 0.6 \neq 1 \]
\[ \neq X \]

**Law of Contradiction (Fuzzy)**

\[ A \cap \overline{A} \neq \emptyset \]

Let \( \mu_A(x) = 0.4 \), \( \therefore \mu_{\overline{X}}(x) = 0.6 \)

so \( \mu_{A \cap B}(x) = \min[\mu_A(x), \mu_{\overline{X}}(x)] \)
\[ = \min[0.4, 0.6] \]
\[ = 0.4 \neq 0 \]
\[ \neq \emptyset \]
Alternative Characteristic Descriptors (Crisp Sets)

\[ \chi_{A \cap B}(x) = \min \left[ \chi_A(x), \chi_B(x) \right] \]
\[ \chi_{A \cup B}(x) = \max \left[ \chi_A(x), \chi_B(x) \right] \]
\[ \chi_{\overline{A}}(x) = 1 - \chi_A(x) \]
\[ \chi_{A-B}(x) = \chi_{A \cap B}(x) = \min \left[ \chi_A(x), 1 - \chi_B(x) \right] \]

Some Other Fuzzy Operators

Yager Class of Fuzzy Complements

\[ \mu_{\overline{A}}(x) = \left(1 - \mu_A(x)^n\right)^{\frac{1}{w}} \]

Algebraic

\[ \mu_{\overline{A}}(x) = 1 - \mu_A(x) \]
\[ \mu_{A \cup B}(x) = \mu_A(x) + \mu_B(x) - \mu_A(x) \cdot \mu_B(x) \]
\[ \mu_{A \cap B}(x) = \mu_A(x) \cdot \mu_B(x) \]

Bounded

\[ \mu_{\overline{A}}(x) = 1 - \mu_A(x) \]
\[ \mu_{A \cup B}(x) = \min \left[ 1, \mu_A(x) + \mu_B(x) \right] \]
\[ \mu_{A \cap B}(x) = \max \left[ 0, \mu_A(x) + \mu_B(x) - 1 \right] \]

Drastic

\[ \mu_{\overline{A}}(x) = 1 - \mu_A(x) \]
\[ \mu_{A \cup B}(x) = \begin{cases} \mu_A(x) \text{ when } \mu_B(x) = 0 \\ \mu_B(x) \text{ when } \mu_A(x) = 0 \\ 1 \text{ otherwise} \end{cases} \]
\[ \mu_{A \cap B}(x) = \begin{cases} \mu_A(x) \text{ when } \mu_B(x) = 1 \\ \mu_B(x) \text{ when } \mu_A(x) = 1 \\ 0 \text{ otherwise} \end{cases} \]