Integration

Why is the integral called an INDEFINITE integral?

It is called indefinite since the solution (because \( \int f(x) \, dx = F(x) + C \) where \( F'(x) = f(x) \)) contains the term + C. We saw the justification of this added term when we first looked at antiderivatives.

The figure below shows the graph of \( \int 2x \, dx = x^2 + C \) where \( \frac{d}{dx}(x^2 + C) = 2x \). This family of curves, positioned as they are because of the value of C, all have the slope of the tangent lines at \( x = 1 \) is the same. So, therefore, we cannot say which one of the family is definitely the solution.

How can we identify definitely which one is the solution?

If we are given some initial information regarding the solution we can do this. For example, find

\[
f(x) = \int 2x \, dx \text{ if } f(0) = 5.
\]

so \( f(x) = x^2 + C \). Since \( f(0) = 5 \), we have and solving for \( C \), \( C = 5 \). \( \therefore f(x) = x^2 + 5 \) is the top curve in the figure. This type of problem is called an **initial value problem**. It allows us, with initial information, to solve for the constant \( C \), a definitely which of the family of curves is the solution.

See Dawkins' Example 4 in the section

- Computing Indefinite Integrals

However, this is NOT called the DEFINITE integral!

The **definite** integral will always yield a numeric solution. The **definite** integral answers the question: Find the area under the curve \( f(x) = \cos x \) from \( x = \frac{\pi}{4} \) to \( x = \frac{\pi}{3} \). (See the figure below)
To calculate the area, we use the fact that the **definite integral** is calculated by finding the indefinite integral and the evaluating the solution at the endpoints as follows.

\[
\int_{x = \frac{\pi}{4}}^{x = \frac{\pi}{3}} \cos x \, dx = -\sin x \bigg|_{x = \frac{\pi}{3}}^{x = \frac{\pi}{4}} = -\sin \frac{\pi}{3} - (-\sin \frac{\pi}{4}) = \left(-\frac{1}{2}\sqrt{3}\right) - \left(-\frac{1}{2}\right) = 2 - \frac{1}{2}\sqrt{3} \approx 0.1589.
\]

Notice, we

- Find an indefinite integral for the function
- Evaluate an indefinite integral at the desired boundary points
- Subtract those values (The term C can be ignored since it will be subtracted off)

That is, \[ \int_{a}^{b} f(x) \, dx = F(b) - F(a) \] where \( F'(x) = f(x) \). This idea of the calculation of the definite integral is part of the **Fundamental Theorem of Calculus** - more about this later.

Be careful though! \[
\int_{x = 0}^{x = 2\pi} \sin x \, dx = \cos x \bigg|_{x = 0}^{x = 2\pi} = (\cos 2\pi) - (\cos 0) = 1 - 1 = 0
\]

As we can see from the figure on the right the area is not 0! The definite integral has a value of zero, but when the area concerned dips below the x-axis, it has a negative affect - since in summing the area, the \( f(x) \) values will be negative. **The definite integral is not always the area.** We have just answered this question-

\[
\int_{x = 0}^{x = 2\pi} \sin x \, dx = 0 .
\]

To obtain the area, we would must know the geometry of the situation and calculate

\[
\text{AREA} = \int_{x = 0}^{x = \pi} \sin x \, dx + \int_{x = \pi}^{x = 2\pi} \sin x \, dx = 1 + |-1| = 2 .
\]

See Dawkins' on-line text, the sections entitled

- Area Problem
- Definition of the Definite Integral
- Computing Definite Integrals